

# HW 10 SOL<sup>n</sup> 5

## SECTION 3.6

①

$$x' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} x$$

$$\text{DET}(A - \lambda I) = \text{DET} \left( \begin{pmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix} \right) = (-1-\lambda)(-1-\lambda) + 4 = 0$$

$$= 5 + \lambda^2 + 2\lambda = 0$$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2} = -1 \pm 2i$$

FIND E-VECTOR ASSOCIATED WITH  $\lambda = -1 + 2i$  :

SOLVE :

$$(A - (-1 + 2i)I)v = 0 \quad \text{FOR } v:$$

$$\begin{pmatrix} -1+1-2i & -4 \\ 1 & -1+1-2i \end{pmatrix} v = 0$$

$$0 = \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rightarrow \begin{pmatrix} -2i & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow v_1 - 2i v_2 = 0 \Rightarrow v_1 = 2i v_2 \quad \text{LET } v_2 = 1$$

$$\Rightarrow v_1 = 2i$$

$$x_1 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

$$\vec{x}_{\text{gen}} = c_1 \text{Re} \left[ e^{t} e^{2it} \begin{pmatrix} 2i \\ 1 \end{pmatrix} \right] + c_2 \text{Im} \left[ e^{t} e^{2it} \begin{pmatrix} 2i \\ 1 \end{pmatrix} \right] \quad \sim$$

1

$$e^{2it} = \cos(2t) + i \sin(2t)$$

$$(\cos(2t) + i \sin(2t)) \begin{pmatrix} 2i \\ 1 \end{pmatrix} = \begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + i \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{x}_1 = e^{-t} \begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix}, \quad \vec{x}_2 = e^{-t} \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}}$$

$$\boxed{\vec{x}_{\text{gen}} = c_1 \vec{x}_1 + c_2 \vec{x}_2}$$

2

$$\vec{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \vec{x}$$

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(-3-\lambda) + 5 = 0 \\ &= \lambda^2 + 2\lambda + 2 = 0 \end{aligned}$$

$\Rightarrow$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = -1 \pm i$$

Solve:

$$(A - (-1+i)I) \vec{v} = 0 \quad \text{FOR E-VECTOR } \vec{v}.$$

$$\begin{pmatrix} 1+1-i & -1 \\ 5 & -3+1-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$\Rightarrow$

$$\begin{pmatrix} 2-i & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

AFTER ROW REDUCTION

$$\Rightarrow (2-i)v_1 - v_2 = 0$$

$$(2-i)v_1 = v_2$$

$$\text{LET } v_1 = 1 \Rightarrow v_2 = 2-i$$

2

$$\xi_1 = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} \quad \vec{x}_{gen} = c_1 \cdot \operatorname{Re} \left[ e^{-t} e^{it} \begin{pmatrix} 1 \\ 2-i \end{pmatrix} \right] + c_2 \cdot \operatorname{Im} \left[ e^{-t} e^{it} \begin{pmatrix} 1 \\ 2-i \end{pmatrix} \right]$$

$$e^{it} = \cos(t) + i \sin(t)$$

$$\Rightarrow e^{it} \begin{pmatrix} 1 \\ 2-i \end{pmatrix} = \left[ \cos(t) + i \sin(t) \right] \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

$$= \begin{pmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} + i \begin{pmatrix} \sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix}$$

$$\therefore \vec{x}_1 = e^{-t} \begin{pmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix}, \quad \vec{x}_2 = e^{-t} \begin{pmatrix} \sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix}$$

$$\vec{x}_{gen} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

### SECTION 3.7

1  $x' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} x$   $\det(A - \lambda I) = (3 - \lambda)(-1 - \lambda) + 4 = 0$   
 $= +1 - 2\lambda + \lambda^2 = 0$   
 $\Rightarrow (\lambda - 1)^2 = 0$   
 $\Rightarrow \lambda = 1$  (multiplicity 2)

Solve:

$$(A - I)v = 0 \text{ For } v:$$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = 2v_2$$

$$\Rightarrow \xi_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

① NOW WE NEED A 2<sup>nd</sup>, LINEARLY INDEPENDENT E-VECTOR  $\xi_2$ .

FIRST SOLVE  $(A-I)^2 v = 0$  :  $(A-I)(A-I) =$   ~~$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$~~

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow$

$(A-I)^2 v = 0$  IS SOLVED BY ANY  $v$  ( $v_1, v_2$  ARE FREE)

CHOOSE  $v_1, v_2$  S.T.  $v_1 \neq 2v_2$  (THAT WOULD JUST GIVE  $\xi_1$ )

ANYTHING WILL WORK HERE.  $\left[ \begin{array}{l} \text{e.g. TAKE } v_1 = 1 \\ v_2 = 0 \end{array} \right.$  THEN  $(A-I) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0$

$$\text{AND } \xi_2 = (I + (A-\lambda)I) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2t \\ 0+t \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{x}_{\text{gen}} = c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1+2t \\ t \end{pmatrix}}$$

$$\textcircled{2} \quad x' = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} x$$

$$\text{DET}(A-\lambda I) = \left(-\frac{3}{2}-\lambda\right)\left(-\frac{1}{2}-\lambda\right) + \frac{1}{4} = 0$$

$$= \frac{3}{4} + 2\lambda + \lambda^2 + \frac{1}{4} = 0$$

$$\Rightarrow \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0$$

$$\Rightarrow \lambda = -1$$

SOLVE  $(A+I)v = 0$  :

$$\begin{pmatrix} -\frac{3}{2}+1 & 1 \\ -\frac{1}{4} & -\frac{1}{2}+1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \begin{aligned} -\frac{1}{2}v_1 + v_2 &= 0 \\ \frac{1}{2}v_1 &= v_2 \end{aligned}$$

$$\text{LET } v_1 = 2 \Rightarrow v_2 = 1.$$

(2)

$$\Rightarrow \xi_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

NOW SOLVE  $(A+I)^2 v = 0$  FOR  $v$ .

$$(A+I)(A+I) = \begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow (A+I)^2 v = 0$  FOR ANY NONZERO  $v$ .  $\Rightarrow v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$   $v_1, v_2$  ARE FREE

TAKE  $\frac{1}{2}v_1 \neq v_2$  SO THAT  $\xi_1, \xi_2$  WILL BE LINEARLY INDEPENDENT.

e.g. TAKE  $v_2 = 1, v_1 = 0$ .

$$\begin{aligned} \text{THEN } \xi_2 &= (I + (A+I)) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} t \\ 1 + \frac{1}{2}t \end{pmatrix} \end{aligned}$$

$$\Rightarrow \boxed{\vec{x}_{gen} = c_1 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} t \\ 1 + \frac{1}{2}t \end{pmatrix}}$$

### SECTION 5.1

① SOLVE :  $y'' + 2y = 0, \quad y'(0) = 1, \quad y'(\pi) = 0.$

$$y = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t)$$

$$\Rightarrow y' = \sqrt{2} c_1 (-\sin(\sqrt{2}t)) + \sqrt{2} c_2 \cos(\sqrt{2}t)$$

$$y'(0) = \sqrt{2} c_2 = 1 \Rightarrow c_2 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$y'(\pi) = -\sqrt{2} c_1 \sin(\sqrt{2}\pi) + \cos(\sqrt{2}\pi) = 0$$

$$\Rightarrow c_1 = \frac{1}{\sqrt{2}} \cot(\sqrt{2}\pi) = \frac{\sqrt{2}}{2} \cot(\sqrt{2}\pi)$$



①

∴

$$y = \frac{\sqrt{2}}{2} \left[ \cot(\sqrt{2}\pi) \cdot \cos(\sqrt{2}t) + \sin(\sqrt{2}t) \right]$$

②

$$y'' + y = 0 \quad y'(0) = 1, \quad y(L) = 0$$

$$y = c_1 \cos(t) + c_2 \sin(t) \Rightarrow y' = -c_1 \sin(t) + c_2 \cos(t)$$

$$\Rightarrow y'(0) = 1 = c_2$$

$$\Rightarrow y = c_1 \cos(t) + \sin(t)$$

$$y(L) = c_1 \cos(L) + \sin(L) = 0$$

$$\Rightarrow c_1 = -\tan(L)$$

∴

$$y(t) = -\tan(L) \cdot \cos(t) + \sin(t)$$

③

$$y'' + 2y = x, \quad y(0) = 0, \quad y(\pi) = 0.$$

$$y(x) = y_{\text{Hom}}(x) + y_p(x), \quad y_{\text{Hom}} = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)$$

$$y_p(x) = Ax + B$$

$$\Rightarrow 0 + 2(Ax + B) = x \Rightarrow A = \frac{1}{2}, B = 0.$$

$$\Rightarrow y(x) = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x) + \frac{1}{2}x$$

$$y(0) = c_1 + 0 + 0 = 0 \Rightarrow c_1 = 0.$$

$$y(\pi) = c_2 \sin(\sqrt{2}\pi) + \frac{\pi}{2} = 0 \Rightarrow c_2 = \frac{\pi}{2 \sin(\sqrt{2}\pi)}$$

∴

$$y(x) = \left( \frac{\pi}{2 \sin(\sqrt{2}\pi)} \right) \cdot \sin(\sqrt{2}x) + \frac{x}{2}$$